Midterm test for Kwantumfysica 1 - 2006-2007

Friday 9 March 2007, 10:15 - 11:00
Werkcollege-zalen group 1 and 2

READ THIS FIRST:

- Clearly write your name and study number on each answer sheet that you use.
- On the first answer sheet, write clearly the total number of answer sheets that you turn in.
- Note that this test has 2 questions, it continues on the backside of the paper!
- Start each question (number 1, 2) on a new answer sheet.
- The test is open book within limits. You are allowed to use the book by Liboff, and one A4 sheet with notes, but nothing more than this.
- If you get stuck on some part of a problem for a long time, it may be wise to skip it and try the next part of a problem first. The test is only 45 minutes.

Useful formulas and constants:

Electron mass	m_e	$= 9.1 \cdot 10^{-31}$ kg
Electron charge	$-e$	$= -1.6 \cdot 10^{-19}$ C
Planck's constant	h	$= 6.626 \cdot 10^{-34}$ Js $= 4.136 \cdot 10^{-15}$ eVs
Planck's reduced constant	$\hbar$	$= 1.055 \cdot 10^{-34}$ Js $= 6.582 \cdot 10^{-16}$ eVs

Problem T1

The position x of a particle is at some time $t = 0$ described by the normalized, real-valued wavefunction

$$\Psi(x) = A(b - |x|)\,, \qquad \text{for } -1\text{ nm} < x < 1\text{ nm}$$
$$\Psi(x) = 0\,, \qquad \text{for } x < -1\text{ nm and } x > 1\text{ nm}$$

with $b = 1$ nm and $A = \sqrt{3/2}$ nm$^{-3/2}$.

a) Make a sketch of both $\Psi(x)$ and the probability density $W(x)$ for the particle's position.

b) Show that the state is normalized for $A = \sqrt{3/2}$ nm$^{-3/2}$, and explain the unit of A.

c) What is the expectation value $\langle \hat{x} \rangle$ for the particle's position at time $t = 0$? Support the answer by showing a calculation, also when you can guess the answer.

d) What is the expectation value $\langle \hat{p}_x \rangle$ for the particle's momentum at time $t = 0$? Support the answer by showing a calculation, also when you can guess the answer.

e) With the particle's wavefunction in this state, you plan to measure the position x at time $t = 0$. What is the probability for detecting a value in the range 0.5 nm $< x <$ 4.5 nm?

Z.O.Z.

Problem T2

For this problem, you must write up your answers in Dirac notation.

Consider a system with a time-independent Hamiltonian

$$\hat{H} = \hat{T} + \hat{V},$$

where T a kinetic-energy term and V a potential-energy term. With respect to a lowest point in the potential, defined as $V = 0$ J, the lowest three energy eigenstates of the system are defined by

$$\begin{aligned} \hat{H}|\varphi_1\rangle &= E_1|\varphi_1\rangle \\ \hat{H}|\varphi_2\rangle &= E_2|\varphi_2\rangle \\ \hat{H}|\varphi_3\rangle &= E_3|\varphi_3\rangle \end{aligned},$$

where $E_1 < E_2 < E_3$ the three energy eigenvalues, and $|\varphi_1\rangle$, $|\varphi_2\rangle$ and $|\varphi_3\rangle$ three orthogonal, normalized energy eigenvectors. The observable $\hat{A}$, is associated with the electric dipole A of this quantum system. For this system,

$$\langle\varphi_1|\hat{A}|\varphi_1\rangle = 0 \quad , \qquad \langle\varphi_2|\hat{A}|\varphi_2\rangle = 2A_0 \quad , \qquad \langle\varphi_3|\hat{A}|\varphi_3\rangle = 3A_0 \ ,$$

$$\langle\varphi_n|\hat{A}|\varphi_m\rangle = \langle\varphi_m|\hat{A}|\varphi_n\rangle = A_0 \ , \quad \text{for all cases} \ \ n \neq m.$$

Note that the states $|\varphi_1\rangle$, $|\varphi_2\rangle$ and $|\varphi_3\rangle$ are energy eigenvectors, and that they are ***not*** eigen vectors of $\hat{A}$

a) What can you say about the possible values of E_1? Discuss the sign, whether it can be zero, and a typical magnitude that you can expect.

b) At some time, the state of the system is (with all c_n a complex-valued constant)

$$|\Psi_S\rangle = c_1|\varphi_1\rangle + c_2|\varphi_2\rangle + c_3|\varphi_3\rangle = \frac{2i}{3}|\varphi_1\rangle + \frac{1}{3}|\varphi_2\rangle + \frac{-2i}{3}|\varphi_3\rangle \ .$$

Show that this is a normalized state.

c) What is for this state $|\Psi_S\rangle$ the expectation value $\langle\hat{A}\rangle$ for A, expressed in A_0?

d) At some other time, defined as $t = 0$, the normalized state of the system is (with again all c_n a complex-valued constant)

$$|\Psi_0\rangle = c_1|\varphi_1\rangle + c_2|\varphi_2\rangle + c_3|\varphi_3\rangle = \frac{\sqrt{5}}{3}|\varphi_1\rangle + \frac{2i}{3}|\varphi_2\rangle + 0\,|\varphi_3\rangle \ .$$

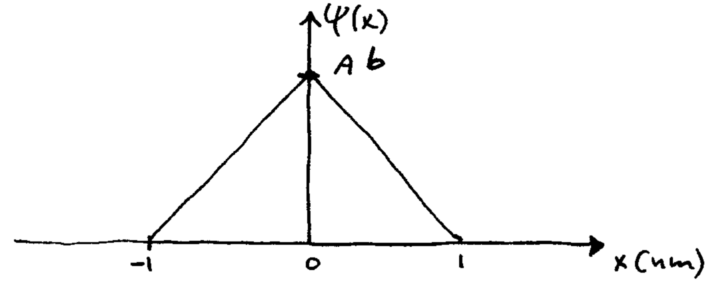
Show that as a function of time $t > 0$, the expectation value for $\langle\hat{A}\rangle$ has oscillations at one frequency only. Determine this frequency and the oscillation amplitude (expressed in the constants that are mentioned above), for the case that the system is in $|\Psi_0\rangle$ at $t = 0$. Use the time-evolution operator (with $\hbar = h/2\pi$)

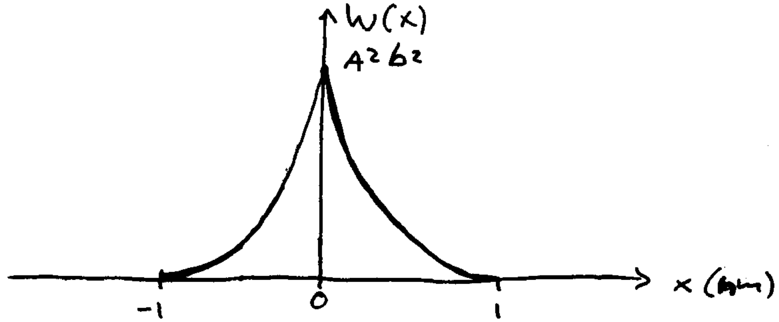
$$\hat{U} = e^{\frac{-i\hat{H}t}{\hbar}} \ .$$

Antwoorden midterm toets Kwantum fysica 1
9 maart 2007

T1

a)





b) Normalized if $\int_{-\infty}^{\infty} |\Psi(x)|^2 \, dx = 1$

$$\int_{-\infty}^{\infty} |\Psi(x)|^2 dx = \int_{-1\,nm}^{1\,nm} A^2 (b-|x|)^2 dx = 2A^2 \int_0^{1\,nm} (b-x)^2 dx = 2A^2 \left[b^2 x - bx^2 + \tfrac{1}{3}x^3 \right]_0^{1\,nm}$$

$$= 2A^2 \left(b^2 - b^2 + \tfrac{1}{3} \right) nm^3 = \tfrac{2}{3} A^2 \cdot nm^3 = 1 \Rightarrow$$

$$A^2 = \frac{3}{2} \frac{1}{nm^3} \Rightarrow A = \sqrt{\tfrac{3}{2}} \, nm^{-3/2}$$

c) $$\langle \hat{x} \rangle = \int_{-\infty}^{\infty} \Psi^*(x) \, x \, \Psi(x) \, dx = \int_{-1\,nm}^{0} A(b+x) \, x \, A(b+x) \, dx + \int_0^{1\,nm} A(b-x) \, x \, A(b-x) \, dx$$

(use $x' = -x$, $dx' = -dx$ $\Rightarrow$)

$$= -\int_0^{1\,nm} A(b-x') \, x' \, A(b-x') \, dx' + \int_0^{1\,nm} A(b-x) \, x \, A(b-x) \, dx$$

$$= 0$$

d) $$\langle \hat{p}_x \rangle = \int_{-\infty}^{\infty} \Psi^*(x) \, \hat{p}_x \, \Psi(x) \, dx = \int_{-\infty}^{\infty} \Psi^*(x) \left(-i\hbar \frac{\partial}{\partial x} \right) \Psi(x) \, dx$$

$$= \int_{-1\,nm}^{0} A(b+x) \left(-i\hbar \frac{\partial}{\partial x} \right) \left(A(b+x) \right) dx + \int_0^{1\,nm} A(b-x) \left(-i\hbar \frac{\partial}{\partial x} \right) \left(A(b-x) \right) dx$$

$$= \int_{-1\,nm}^{0} A(b+x)(-i\hbar) \, A \, dx + \int_0^{1\,nm} A(b-x)(+i\hbar) A \, dx = 0$$

e) $$P(0.5\,nm < x < 1.5\,nm) = \int_{0.5\,nm}^{1.5\,nm} W(x) \, dx$$

$$= \int_{0.5\,nm}^{1\,nm} |\Psi(x)|^2 \, dx = \int_{0.5\,nm}^{1\,nm} A^2 (b-x)^2 \, dx$$

$$= A^2 \left[b^2 x - bx^2 + \tfrac{1}{3} x^3 \right]_{0.5\,nm}^{1\,nm}$$

$$= A^2 \left(\tfrac{1}{3}(1\,nm^3) - \left(\tfrac{1}{2}\,nm^3 \right) + \left(\tfrac{1}{4}\,nm^3 \right) - \left(\tfrac{1}{24}\,nm^3 \right) \right)$$

$$= \left(\tfrac{3}{2}\,nm^{-3} \right) \left(\left(\tfrac{8}{24} - \tfrac{12}{24} + \tfrac{6}{24} - \tfrac{1}{24} \right) nm^{+3} \right)$$

$$= \frac{3}{2} \cdot \frac{1}{24} = \frac{1}{16}$$

T2 a) The system has for its lowest energy eigenstates a set of 3 discrete states $\Rightarrow$ these correspond to localized (= confined) states, where the particle is trapped near the minimum in the potential energy V (that is, these states are not a free-particle state) $\Rightarrow$ $\langle \hat{V} \rangle$ and $\langle T \rangle$ must both be larger than 0 J. The particle will have some zero-point energy in the state $|\varphi_1\rangle$, so $E_1 > 0\,J$

The typical magnitude of E_1 is harder to estimate here, but for the ground state $E_1 \approx \langle \hat{T} \rangle \approx \langle \hat{V} \rangle$

b) We need to show that $\langle \Psi_s | \Psi_s \rangle = 1$, and can use that

$\langle \varphi_n | \varphi_m \rangle = 1$ for $n = m$
$\langle \varphi_n | \varphi_m \rangle = 0$ for $n \neq m$

$$\langle \Psi_s | \Psi_s \rangle = \left(c_1^* \langle \varphi_1 | + c_2^* \langle \varphi_2 | + c_3^* \langle \varphi_3 |\right)\left(c_1 |\varphi_1\rangle + c_2 |\varphi_2\rangle + c_3 |\varphi_3\rangle\right)$$

$$= c_1^* c_1 \langle \varphi_1 | \varphi_1 \rangle + c_2^* c_2 \langle \varphi_2 | \varphi_2 \rangle + c_3^* c_3 \langle \varphi_3 | \varphi_3 \rangle$$

$$= \left|\tfrac{2}{3}\right|^2 + \left|\tfrac{1}{3}\right|^2 + \left|\tfrac{2}{3}\right|^2 = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = 1$$

c) $\langle \hat{A} \rangle = \langle \Psi_s | \hat{A} | \Psi_s \rangle = \left(c_1^* \langle \varphi_1 | + c_2^* \langle \varphi_2 | + c_3^* \langle \varphi_3 |\right) \hat{A} \left(c_1 |\varphi_1\rangle + c_2 |\varphi_2\rangle + c_3 |\varphi_3\rangle\right)$

$$= c_1^* c_1 \langle \varphi_1 | \hat{A} | \varphi_1 \rangle + c_1^* c_2 \langle \varphi_1 | \hat{A} | \varphi_2 \rangle + c_1^* c_3 \langle \varphi_1 | \hat{A} | \varphi_3 \rangle +$$
$$c_2^* c_1 \langle \varphi_2 | \hat{A} | \varphi_1 \rangle + c_2^* c_2 \langle \varphi_2 | \hat{A} | \varphi_2 \rangle + c_2^* c_3 \langle \varphi_2 | \hat{A} | \varphi_3 \rangle +$$
$$c_3^* c_1 \langle \varphi_3 | \hat{A} | \varphi_1 \rangle + c_3^* c_2 \langle \varphi_3 | \hat{A} | \varphi_2 \rangle + c_3^* c_3 \langle \varphi_3 | \hat{A} | \varphi_3 \rangle$$

$$= c_1^* c_1 (0) + c_2^* c_2 (2A_0) + c_3^* c_3 (3A_0)$$
$$+ \left(\cancel{c_1^* c_2} + c_1^* c_3 + \cancel{c_2^* c_1} + \cancel{c_2^* c_3} + c_3^* c_1 + \cancel{c_3^* c_2}\right) A_0$$

$$= \frac{2}{9} A_0 + \frac{4}{9} 3 A_0 + \left(2\left(\frac{2}{3} \cdot -\frac{2}{3}\right)\right) A_0 = \left(\frac{2}{9} + \frac{12}{9} - \frac{8}{9}\right) A_0 = + \frac{6}{9} A_0 = \frac{2}{3} A_0$$

d) $\langle \hat{A} \rangle(t) = \langle \Psi(t) | \hat{A} | \Psi(t) \rangle$ with $|\Psi(t)\rangle = \hat{U} |\Psi_0\rangle = e^{-i\hat{H}t/\hbar} |\Psi_0\rangle$
and $\langle \Psi(t) | = \langle \Psi_0 | \hat{U}^\dagger = \langle \Psi_0 | e^{+i\hat{H}t/\hbar}$

$$\Rightarrow \langle \hat{A} \rangle(t) = \langle \Psi_0 | \hat{U}^\dagger \hat{A} \hat{U} | \Psi_0 \rangle$$

$$= \left(c_1^* e^{+iE_1 t/\hbar} \langle \varphi_1 | + c_2^* e^{+iE_2 t/\hbar} \langle \varphi_2 |\right) \hat{A} \left(c_1 e^{-iE_1 t/\hbar} |\varphi_1\rangle + c_2 e^{-iE_2 t/\hbar} |\varphi_2\rangle\right)$$

$$= |c_1|^2 \langle \varphi_1 | \hat{A} | \varphi_1 \rangle + |c_2|^2 \langle \varphi_2 | \hat{A} | \varphi_2 \rangle + \left(c_1^* c_2 e^{\frac{i(E_1 - E_2)t}{\hbar}} + c_2^* c_1 e^{\frac{i(E_2 - E_1)t}{\hbar}}\right) \langle \varphi_1 | \hat{A} | \varphi_2 \rangle$$

$$= \frac{5}{9} \cdot 0 + \frac{4}{9} 2A_0 + \frac{i 2\sqrt{5}}{9} \left(e^{-i(E_2 - E_1)t/\hbar} - e^{+i(E_2 - E_1)t/\hbar}\right) A_0$$

$$= \frac{8}{9} A_0 + \frac{4}{9}\sqrt{5} \sin\left(\left(\frac{E_2 - E_1}{\hbar}\right) t\right) A_0$$

Amplitude is $\frac{4}{9}\sqrt{5}\, A_0$

Only frequency = $f = \frac{E_2 - E_1}{2\pi\hbar}$, $\omega = 2\pi f$